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Phase transition in a model with non-compact symmetry on Bethe lattice and the replica limit

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Abstract. We solve the $O(n, 1)$ nonlinear vector model on the Bethe lattice and show that it exhibits a transition from ordered to disordered state for $0 \leqslant n < 1$. If the replica limit $n \to 0$ is taken carefully, the model is shown to reduce to the corresponding supersymmetric model. The latter was introduced by Zirnbauer as a toy model for the Anderson localization transition. We argue thus that the non-compact replica models describe correctly the Anderson transition features. This should be contrasted to their failure in the case of the level correlation problem.

1. Introduction

In the study of systems with quenched randomness, historically the first way of treating the disorder averages was the so-called replica method of Edwards and Anderson [1]. In this method one introduces n 'replicas' of the original system and calculates annealed averages for this replicated system. Then the use of the identity

$$\overline{\log Z} = \lim_{n\to 0} \frac{\overline{Z^n} - 1}{n}$$

allows one to recover the properties of the original system.

For the problem of Anderson localization, the replica fields describing electrons may be chosen to be either fermionic (Grassmann) or bosonic. In either case one can construct an effective field theoretic description in terms of a nonlinear σ-model for the matrix field which belongs to some compact manifold for fermionic replicas and some non-compact manifold for bosonic ones (see, e.g. [2]).

Another way to treat the disorder is the supersymmetry method of Efetov [3]. In this method one introduces both fermionic and bosonic degrees of freedom and the resulting σ-model field is of the supermatrix structure.

It was realized some time ago that the replica method suffers from serious drawbacks. Verbaarschot and Zirnbauer showed explicitly [4] that the replica method fails to give a correct non-perturbative result for a problem of energy-level correlation which is equivalent to a zero-dimensional σ-model, while the supersymmetry method works nicely. Since then the prevailing opinion in the literature on the Anderson localization problem has been that

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the replica method may be at best considered as a perturbative tool, not being able to describe properties of disordered (localized) phase.

The aim of our paper is to reconsider the correspondence between supersymmetric models and non-compact models in the replica limit. For this purpose we analyse the solution of two 'toy' models on the Bethe lattice. The first is one of the simplest models with non-compact symmetry, namely the $O(n,1)/O(n)$-vector model. The other is its supersymmetric counterpart, the so-called 'hyperbolic superplane' (see [5, 6]).

The paper is organized as follows. In section 2 we set up the general description of properties of the $O(n,1)$-model in terms of a single 'distribution function of local-order parameter' $P(\theta)$. Doing this in the spirit of dimensional continuation, we consider parameter n to be any real number large enough to ensure convergence of integrals. In section 3 we similarly consider a supersymmetric version of our model. Section 4 is the central section of the paper; here we discuss the replica limit $n \to 0$ and show that if we take it carefully all the results for the $O(n,1)$ model *exactly* reproduce results of the supersymmetric treatment. In the following sections we proceed to solve the $O(n,1)$ model. Our analysis follows very closely that of previous papers devoted to the problem of Anderson localization on the Bethe lattice [7–9] and therefore we omit many details. We find that the $O(n,1)$ model exhibits two phases with a phase transition between them for any $0 \leqslant n < 1$. We obtain the critical behaviour of different correlators near this transition and show that it is exactly the same as exhibited by the supersymmetric model. Finally, section 8 contains a discussion of our results.

2. $O(n,1)$-model: general equations

We start with the Hamiltonian

$$\mathcal{H} = J \sum_{\langle ij \rangle} \boldsymbol{n}_i \cdot \boldsymbol{n}_j + H \sum_i \sigma_i .$$

Here i and j refer to the sites of the Bethe lattice with coordination number $m+1$ and $\boldsymbol{n} = (\sigma, \boldsymbol{\pi})$ is a $(n+1)$-component vector sweeping the hyperboloid $H^{n,1}$, defined by $\boldsymbol{n}^2 = \sigma^2 - \boldsymbol{\pi}^2 = 1$. This hyperboloid is the symmetric space associated with the $O(n,1)$ group: $H^{n,1} = O(n,1)/O(n)$. We parametrize $\boldsymbol{n}$ as follows $\sigma = (1+\boldsymbol{\pi}^2)^{1/2} = \cosh\theta$, $0 \leqslant \theta < \infty$,

$$\boldsymbol{\pi} = \begin{pmatrix} \sinh\theta\cos\phi_1 \\ \sinh\theta\sin\phi_1\cos\phi_2 \\ \vdots \\ \sinh\theta\sin\phi_1\ldots\sin\phi_{n-1} \end{pmatrix} \qquad \phi_1,\ldots,\phi_{n-2} \in [0,\pi] \qquad \phi_{n-1} \in [0, 2\pi).$$

With this parametrization the scalar product is $\boldsymbol{n}_i \cdot \boldsymbol{n}_j = \sigma_i\sigma_j - \boldsymbol{\pi}_i \cdot \boldsymbol{\pi}_j \geqslant \cosh(\theta_i - \theta_j) \geqslant 1$ and therefore, the Hamiltonian $\mathcal{H}$ is bounded from below only for $J, H \geqslant 0$. The $O(n,1)$-invariant measure on $H^{n,1}$ is

$$\mathrm{d}\boldsymbol{n} = a\,\mathrm{d}\theta\,\sinh^{n-1}\theta\,\mathrm{d}\phi_1\,\sin^{n-2}\phi_1\,\mathrm{d}\phi_2\,\sin^{n-3}\phi_2\ldots\mathrm{d}\phi_{n-1}$$

where a is a normalization constant to be fixed later.

Now we introduce a 'distribution function of the local-order parameter' $P(\boldsymbol{n})$ in the usual manner. Namely, we cut one of the $m+1$ branches coming from site $\boldsymbol{n}$ and integrate the part of the Boltzmann weight $\exp(-\mathcal{H})$ over this branch. The resulting function $P(\boldsymbol{n})$ satisfies the integral equation

$$P(\boldsymbol{n}) = \int \mathrm{d}\boldsymbol{n}'\, L(\boldsymbol{n},\boldsymbol{n}')D(\boldsymbol{n}')P^m(\boldsymbol{n}') \tag{1}$$

where we introduced the following notation: $L(\boldsymbol{n},\boldsymbol{n}') = \exp(-J\boldsymbol{n}\cdot\boldsymbol{n}')$, $D(\boldsymbol{n}') = \exp(-H\sigma')$. Knowledge of the function $P(\boldsymbol{n})$ allows us to calculate the partition function $Z = \int \mathrm{d}\boldsymbol{n}\, D(\boldsymbol{n})P^{m+1}(\boldsymbol{n})$, one-site, averages $\langle A(\boldsymbol{n})\rangle = Z^{-1}\int \mathrm{d}\boldsymbol{n}\, A(\boldsymbol{n})D(\boldsymbol{n})P^{m+1}(\boldsymbol{n})$, and 'weighted' correlators

$$\langle A(\boldsymbol{n}_0)B(\boldsymbol{n}_r)\rangle_w = \frac{N(r)}{Z}\int \mathrm{d}\boldsymbol{n}_0\, A(\boldsymbol{n}_0)D(\boldsymbol{n}_0)P^m(\boldsymbol{n}_0) \times\left(\prod_{i=1}^{r}\int \mathrm{d}\boldsymbol{n}_i\, M(\boldsymbol{n}_{i-1},\boldsymbol{n}_i)\right)P(\boldsymbol{n}_r)B(\boldsymbol{n}_r) \tag{2}$$

$$M(\boldsymbol{n},\boldsymbol{n}') = L(\boldsymbol{n},\boldsymbol{n}')D(\boldsymbol{n}')P^{m-1}(\boldsymbol{n}')$$

where the factor $N(r) = (m+1)m^{r-1}$ counts the number of sites located at the distance r from a given site (without this factor all the correlators exponentially decay because of the geometry of the Bethe lattice).

The constant a in the definition of the measure $\mathrm{d}\boldsymbol{n}$ can be chosen arbitrarily. It is easy to see that rescaling of the measure changes the overall normalization of $P(\boldsymbol{n})$ and Z but does not affect either one-site averages or correlators. This allows us to choose a convenient normalization for $P(\boldsymbol{n})$ as follows. Note that when $H = 0$ equation (1) admits a constant solution. Then we require that this solution be simply $P(\boldsymbol{n}) = 1$ or, equivalently, that

$$\int \mathrm{d}\boldsymbol{n}'\, L(\boldsymbol{n},\boldsymbol{n}') = 1. \tag{3}$$

This fixes $a = (J/2\pi)^p(2K_p(J))^{-1}$, where by p we denote the combination $(n-1)/2$ which is often encountered in the following, and $K_p(J)$ is the modified Bessel function.

The magnetic field H breaks the $O(n,1)$ symmetry down to $O(n)$. Then the function P may depend only on σ or, equivalently, on θ. This allows us to perform angular integrations in (1) yielding

$$P(\theta) = \int_0^\infty \mathrm{d}\theta'\, L_{\mathrm{L}}(\theta,\theta')D(\theta')P^m(\theta') \tag{4}$$

$$L_{\mathrm{L}}(\theta,\theta') = \frac{1}{2K_p(J)}\left(\frac{\sinh\theta'}{\sinh\theta}\right)^p \times\exp(-J\cosh\theta\cosh\theta')(2\pi J\sinh\theta\sinh\theta')^{1/2}I_{(n/2)-1}(J\sinh\theta\sinh\theta')$$

$$D(\theta') = \exp(-H\cosh\theta') \tag{5}$$

where $I_\nu(x)$ is the modified Bessel function. Similar integration may be done in expressions for the partition function $Z = aS_{n-1}\int_0^\infty \mathrm{d}\theta\, \sinh^{n-1}\theta\, D(\theta)P^{m+1}(\theta)$, where $S_{n-1} = 2\pi^{n/2}/\Gamma(n/2)$ is the volume of the sphere S^{n-1}, and correlators. In particular upon averaging of $\boldsymbol{n}$ only the σ-component survives giving the 'order parameter'

$$\langle\sigma\rangle \equiv \langle\cosh\theta\rangle = \frac{aS_{n-1}}{Z}\int_0^\infty \mathrm{d}\theta\, \sinh^{n-1}\theta\cosh\theta\, D(\theta)P^{m+1}(\theta).$$

For the invariant correlator $\langle\boldsymbol{n}_0\cdot\boldsymbol{n}_r\rangle = \langle\sigma_0\sigma_r\rangle - \langle\boldsymbol{\pi}_0\cdot\boldsymbol{\pi}_r\rangle$ the angular integrals give different kernels for longitudinal $G^{\mathrm{L}}(r) \equiv \langle\cosh\theta_0\cosh\theta_r\rangle_w$ and transverse $G^{\mathrm{T}}_{ij}(r) \equiv \langle\pi_{0i}\pi_{rj}\rangle_w$ parts

$$G^{\mathrm{L}}(r) = \frac{aS_{n-1}N(r)}{Z}\int_0^\infty \mathrm{d}\theta_0\, \sinh^{n-1}\theta_0\cosh\theta_0\, D(\theta_0)P^m(\theta_0) \times\left(\prod_{i=1}^{r}\int_0^\infty \mathrm{d}\theta_i\, M_{\mathrm{L}}(\theta_{i-1},\theta_i)\right)P(\theta_r)\cosh\theta_r \tag{6}$$

$$G_{ij}^{\mathrm{T}}(r) = \delta_{ij}\frac{aS_{n-1}N(r)}{nZ}\int_0^\infty \mathrm{d}\theta_0\ \sinh^n\theta_0 D(\theta_0)P^m(\theta_0) \times\left(\prod_{i=1}^r\int_0^\infty \mathrm{d}\theta_i\ M_{\mathrm{T}}(\theta_{i-1},\theta_i)\right)P(\theta_r)\sinh\theta_r \tag{7}$$

where $M_{\mathrm{L,T}}(\theta,\theta') = L_{\mathrm{L,T}}(\theta,\theta')D(\theta')P^{m-1}(\theta')$,

$$L_{\mathrm{T}}(\theta,\theta') = \frac{1}{2K_p(J)}\left(\frac{\sinh\theta'}{\sinh\theta}\right)^p \times\exp(-J\cosh\theta\cosh\theta')(2\pi J\sinh\theta\sinh\theta')^{1/2}I_{n/2}(J\sinh\theta\sinh\theta').$$

Equations (2), (6) and (7) may be written symbolically as

$$\langle A(\boldsymbol{n}_0)B(\boldsymbol{n}_r)\rangle_w = Z^{-1}N(r)\langle A(\boldsymbol{n}_0)|\hat{M}^r|B(\boldsymbol{n}_r)\rangle$$

where $\hat{M}$ represents an integral operator with one of the kernels M, M_{L}, or M_{T}. Introducing the complete set $|\phi_\Lambda\rangle$ of eigenfunctions of $\hat{M}$: $\hat{M}|\phi_\Lambda\rangle = \Lambda|\phi_\Lambda\rangle$ we can rewrite the correlators as

$$\langle A(\boldsymbol{n}_0)B(\boldsymbol{n}_r)\rangle_w = \frac{m+1}{mZ}\sum_\Lambda (m,\Lambda)^r\frac{\langle A|\phi_\Lambda\rangle\langle\phi_\Lambda|B\rangle}{\langle\phi_\Lambda|\phi_\Lambda\rangle}. \tag{8}$$

We choose operators $\hat{M}$ to be non-symmetric, which means that the left and right eigenfunctions are different.

3. Supersymmetric version: hyperbolic superplane

In this section we consider the supersymmetric version of the $O(n,1)$ model, namely a nonlinear model with field-taking values on the so-called hyperbolic superplane. This object is constructed as follows (see [5, 6]). We consider a set of five-component vectors

$$\bar{\psi} = (\sigma,\pi_1,\pi_2,\bar{\xi},-\xi)$$

where the first three components are commuting, whereas the last two are Grassmannians (we use the adjoint of the second kind, see [10] for a review of supermathematics). Next we consider the group G of linear transformations in the space of vectors ψ which preserve the 'length' $\|\psi\|^2 = \sigma^2 - \pi_1^2 - \pi_2^2 - 2\bar{\xi}\xi$. Let K be the subgroup of G which separately preserves σ^2 and $\pi_1^2+\pi_2^2+2\bar{\xi}\xi$. Then the coset space G/K is isomorphic to the space of vectors ψ of unit length $\|\psi\| = 1$. This is the hyperbolic superplane.

We will use the following parametrization of G/K: $\pi_1 = \sinh\theta\cos\phi$, $\pi_2 = \sinh\theta\sin\phi$, $\sigma = (1+\sinh^2\theta + 2\bar{\xi}\xi)^{1/2} = \cosh\theta + \bar{\xi}\xi/\cosh\theta$. In this parametrization the G-invariant measure on G/K is

$$\mathrm{d}\psi = a\frac{1}{\sigma}\,\mathrm{d}\pi_1\,\mathrm{d}\pi_2\,\mathrm{d}\bar{\xi}\,\mathrm{d}\xi = a\left(1-\frac{\bar{\xi}\xi}{\cosh^2\theta}\right)\mathrm{d}\theta\sinh\theta\,\mathrm{d}\phi\,\mathrm{d}\bar{\xi}\,\mathrm{d}\xi.$$

The Hamiltonian in this case is taken to be

$$\mathcal{H} = J\sum_{\langle ij\rangle}\bar{\psi}_i\psi_j + H\sum_i\sigma_i.$$

We again choose the constant a in the definition of $\mathrm{d}\psi$ such that the integral $\int\mathrm{d}\psi'\ \exp(-J\bar{\psi}\psi') = 1$. This gives $a = \mathrm{e}^J/2\pi$. Proceeding as in section 2 we introduce the function $P(\psi)$ (by symmetry it actually depends only on σ) which satisfies the equation

$$P(\sigma) = \int\mathrm{d}\psi'\ \exp(-J\bar{\psi}\psi' - H\sigma')P^m(\sigma').$$

Expanding both the left-hand side and the right-hand side in powers of Grassmann variables and integrating them out we get, from the last equation,

$$P(\theta) = \exp(J - J\cosh\theta)\,\mathrm{e}^{-H} P^m(0) + \int_0^\infty \mathrm{d}\theta'\, L_{\mathrm{L}0}(\theta,\theta') D(\theta') P^m(\theta') \tag{9}$$

$$L_{\mathrm{L}0}(\theta,\theta') = J\sinh\theta\,\exp(J - J\cosh\theta\cosh\theta') I_1(J\sinh\theta\sinh\theta'). \tag{10}$$

The first term in (9) is the boundary term resulting from integration by parts.

If we put $\theta = 0$ in (9) we obtain $P(0) = \mathrm{e}^{-H} P^m(0)$ which means that $P(0) = \exp(H/m-1))$ or $P(0) = 0$. To have $P(\theta) = 1$ as a solution for $H = 0$ we have to choose $P(0) = \exp(H/(m-1))$.

We can also perform Grassmann integrations in formulae for the partition function $Z = \mathrm{e}^J \exp(2H/(m-1))$, one-site averages $\langle A(\sigma)\rangle = A(0)$ and 'longitudinal' correlators $\langle A(\sigma_0)B(\sigma_r)\rangle_w = N(r)A(0)B(0)$. In particular we have

$$\langle\cosh\theta\rangle = 1 \qquad G^{\mathrm{L}}(r) \equiv \langle\sigma_0\sigma_r\rangle_w = N(r) \qquad G^{\mathrm{L}}_{\mathrm{c}}(r) = 0 \tag{11}$$

where subscript c refers to the connected correlator. For the transverse correlator $G^{\mathrm{T}}_{ij}(r) \equiv \langle\pi_{0i}\pi_{rj}\rangle_w$ we obtain after some calculation

$$G^{\mathrm{T}}_{ij}(r) = \delta_{ij}\frac{\mathrm{e}^J N(r)}{Z}\int_0^\infty \mathrm{d}\theta_0\, D(\theta_0)P^m(\theta_0)$$
$$\times\left(\prod_{i=1}^r \int_0^\infty \mathrm{d}\theta_i\, L_{\mathrm{T}0}(\theta_{i-1},\theta_i) D(\theta_i) P^{m-1}(\theta_i)\right) P(\theta_r)\sinh\theta_r \tag{12}$$

$L_{\mathrm{T}0}(\theta,\theta') = J\sinh\theta\,\exp(J - J\cosh\theta\cosh\theta') I_0(J\sinh\theta\sinh\theta')$.

4. Replica limit

Now we study what happens with the equations of section 2 for the $O(n,1)$-model in the replica limit when $n \to 0$, namely whether they reduce to those of section 3 or not.

First of all, if we simply set $n = 0$ in (4) we get $P(\theta) = \int \mathrm{d}\theta'\, L_{\mathrm{L}0}(\theta,\theta') D(\theta') P^{m+1}(\theta')$ with kernel (10), which differs from (9) by the absence of the boundary term. From this we could conclude, in particular, that $P(0) = 0$ in the replica limit and, therefore, this limit gives the incorrect answers. However, this simple recipe is wrong. To see this, let us set $\theta = 0$ in (4) before taking the replica limit. Using the small-z expansion $I_\nu(z) \approx (z/2)^\nu/\Gamma(\nu+1)$ valid for $\nu \neq -1, -2, \ldots$, we get

$$P(0) = aS_{n-1}\int_0^\infty \mathrm{d}\theta\,\sinh^{n-1}\theta\,\exp(-J\cosh\theta) D(\theta) P^m(\theta). \tag{13}$$

If we assume that $P(0) \neq 0$ then the integral in the last equation is divergent at the lower limit if we set $n = 0$. At the same time $aS_{n-1} = 0$ for $n = 0$, so the expression (13) is of the type $0\cdot\infty$ in the replica limit and should be studied in a more careful way. For this purpose, let us consider the following identity:

$$S_{n-1}\int_0^\infty \mathrm{d}\theta\,\sinh^{n-1}\theta f(\theta)$$
$$= f(0)S_{n-1}\int_0^\infty \mathrm{d}\theta\,\sinh^{n-1}\theta + S_{n-1}\int_0^\infty \mathrm{d}\theta\,\sinh^{n-1}\theta[f(\theta) - f(0)].$$

Here the first term is finite for $0 < n < 1$ and gives $\pi^p\Gamma(-p)f(0)$, whereas in the second term the difference $f(\theta) - f(0)$ makes the integral convergent even for $n = 0$. Now we can safely take the replica limit in which the second term disappears, and we get

$$\lim_{n\to 0} S_{n-1} \int_0^\infty \mathrm{d}\theta\ \sinh^{n-1}\theta f(\theta) = f(0). \tag{14}$$

Therefore, the replica limit of (13) is simply $P(0) = \mathrm{e}^{-H} P^m(0)$ which is exactly the result for $P(0)$ from section 3.

Now we can perform a similar trick for arbitrary θ

$$\int_0^\infty \mathrm{d}\theta'\, L_\mathrm{L}(\theta,\theta') f(\theta') = f(0) \int_0^\infty \mathrm{d}\theta'\, L_\mathrm{L}(\theta,\theta') + \int_0^\infty \mathrm{d}\theta'\, L_\mathrm{L}(\theta,\theta')[f(\theta') - f(0)].$$

Before we take the replica limit, the integral in the first term here equals unity due to normalization of kernel $L(\boldsymbol{n},\boldsymbol{n}')$ (3). In the second term we can safely take the replica limit as before. The kernel there becomes $L_{\mathrm{L}0}(\theta,\theta')$ of (10). After that we split the second term again into two pieces to get

$$\lim_{n\to 0} \int_0^\infty \mathrm{d}\theta'\, L_\mathrm{L}(\theta,\theta') f(\theta') = f(0)\left(1 - \int_0^\infty \mathrm{d}\theta'\, L_{\mathrm{L}0}(\theta,\theta')\right) + \int_0^\infty \mathrm{d}\theta'\, L_{\mathrm{L}0}(\theta,\theta') f(\theta').$$

The integral in the first term can be performed (see e.g. [11]) and we get as the result

$$\lim_{n\to 0} \int_0^\infty \mathrm{d}\theta'\, L_\mathrm{L}(\theta,\theta') f(\theta') = f(0)\ \exp(J - J\cosh\theta) + \int_0^\infty \mathrm{d}\theta'\, L_{\mathrm{L}0}(\theta,\theta') f(\theta'). \tag{15}$$

Then in the replica limit (4) becomes exactly (9) of section 3!

The alternative way of getting (15) is to separate the two contributions to the modified Bessel function which enters the integral kernel $L_\mathrm{L}(\theta,\theta')$ using the recursion relation

$$I_{(n/2)-1}(z) = \frac{n}{z} I_{(n/2)}(z) + I_{(n/2)+1}(z) \simeq \frac{1}{\Gamma(n/2)}\left(\frac{z}{2}\right)^{(n/2)-1} + I_1(z) \qquad n \to 0.$$

The crucial point is that the first term here cannot be neglected, although it has a vanishing coefficient in the limit $n \to 0$, since the corresponding integral over θ' will diverge in this limit. Thus, in full analogy with the treatment of (13), this singular contribution should be first evaluated at finite n, and only then can the limit $n \to 0$ be taken. This again gives (15).

The use of prescriptions (14) and (15) allows us to show that equations for all the quantities of interest from section 2 in the replica limit reduce to those of the supersymmetry method of section 3. This is the main result of our analysis. In the remaining sections we find that our model exhibits ordered and disordered phases for $0 \leqslant n < 1$. We find the transition point between them and solve for the critical behaviour of correlators near this transition. We explicitly show then that in the replica limit this behaviour is identical to the one found in [7, 8] for the supersymmetric σ-model and in [9] for the Anderson model on the Bethe lattice.

5. Ordered phase and transition point

For $J \gg 1$ we expect an ordered state with spontaneously broken symmetry, where all the $\boldsymbol{n}$s are slightly fluctuating around σ-direction. The transverse components $\boldsymbol{\pi}$ are small and we can expand in them: $\sigma \approx 1 + \frac{1}{2}\boldsymbol{\pi}^2 + \cdots$. Then the kernel of (1) becomes Gaussian and, therefore, the equation admits (for any value of n) a solution which is also Gaussian

$$P(\boldsymbol{n}) = c\ \exp(-\tfrac{1}{2}\alpha\boldsymbol{\pi}^2) \tag{16}$$

where $\alpha = \{J(m-1) - H + [(J(m-1)-H)^2 + 4mJH]^{1/2}\}/2m$ and c is some normalization constant. With this form of $P(\boldsymbol{n})$ we perform Gaussian integrals to find for a small magnetic field

$$G^{\mathrm{L}}_{\mathrm{c}}(r) \propto nm^{-r} \qquad G^{\mathrm{T}}_{ij}(r) = \delta_{ij}\frac{1}{(m-1)\rho_{\mathrm{s}}}\exp\left(-\frac{H}{(m-1)\rho_{\mathrm{s}}}r\right) \tag{17}$$

where 'spin stiffness' $\rho_{\mathrm{s}} = J$. So the longitudinal correlator is massive, while transverse modes are Goldstones with long-ranged correlations for $H = 0$.

The non-compact nature of the $O(n,1)$ model makes it rather unusual from the traditional point of view. For example, mean-field theory for this model does not give any transition at all: the system is always ordered. If we expect to have disordered phase for $J \ll 1$, then in this phase the order parameter will be large for small magnetic field, diverging when $H \to 0$.

The symmetry breaking factor $D(\theta)$ (5) becomes significant for $\cosh\theta \sim H^{-1}$ or $\theta \sim \ln(2/H)$, therefore it is convenient to change variables to $t = \ln(H\cosh\theta)$ in (4). After such a change the argument of the Bessel function becomes large for small H and, using asymptotic formula $(2\pi z)^{1/2} I_\nu(z) \to \mathrm{e}^z$ we arrive in the limit $H \to 0$ at

$$P(t) = \int_{-\infty}^{\infty} \mathrm{d}t'\, L(t-t')D(t')P^m(t') \tag{18}$$

$$L(t) = \frac{1}{2K_p(J)}\,\exp(-pt - J\cosh t) \qquad D(t) = \exp(-\mathrm{e}^t).$$

This equation was derived under the assumption that the solution P is a function of variable $H\cosh\theta$ for small H. Such a solution becomes $P = 1$ in the limit $H = 0$. The solution of the 'ordered type' (16) is not contained in (18) or, rather, it corresponds to the trivial solution $P = 0$.

The qualitative behaviour of the solution of (18) should be as follows. For large positive t ($t \gg 0$) the function $P(t)$ exponentially goes to zero because of the symmetry breaking term $D(t')$. For large negative t ($t \ll 0$) the main contribution to the integral in (18) comes from $t' \ll 0$ because the kernel $L(t-t')$ is sharply peaked at $t = t'$. But in this region $D(t') \approx 1$ and the equation (18) admits the solution $P = 1$. In the region $t \sim 0$ there should be a kink connecting the two asymptotic soutions. The precise form of the solution can be found numerically by iterations starting with $P = 1$. But such an iterative procedure is convergent to the solution of described type only for small enough J. If $J > J_{\mathrm{c}}$ where J_{c} is the critical coupling, this solution becomes unstable, the kink at $t \sim 0$ starts to move to negative t until the solution $P(\theta)$ takes the form characteristic of the ordered phase. In fact (18) is very similar to the one studied by Zirnbauer [8], so we follow very closely the stability analysis of this paper.

For $t \ll 0$ we drop $D(t')$ and linearize the simplified equation around the constant solution $P(t) = 1 - \delta P(t)$ with

$$\delta P(t) = m\int_{-\infty}^{\infty} \mathrm{d}t'\, L(t-t')\delta P(t'). \tag{19}$$

Because of the translational invariance of this equation it admits exponential solutions $\int_{-\infty}^{\infty} \mathrm{d}t'\, L(t-t')\,\mathrm{e}^{\nu t'} = \Lambda(\nu)\,\mathrm{e}^{\nu t}$, where

$$\Lambda(\nu) = \frac{K_{p+\nu}(J)}{K_p(J)} \tag{20}$$

is an eigenvalue of the integral operator with the kernel $L(t-t')$.

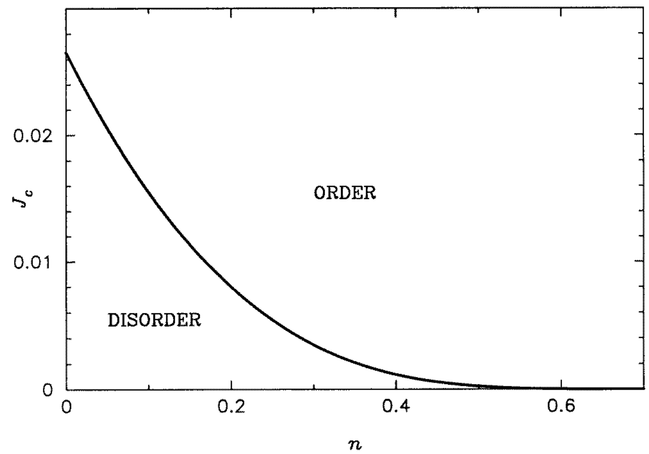

Figure 1. Solution of equation (22) (phase diagram on the $J-n$ plane) for $m=2$.

Let us make a mathematical remark. In terms of vector $\boldsymbol{n}$ (19) corresponds to $\delta P(\boldsymbol{n}) = m\int \mathrm{d}\boldsymbol{n}'\, L(\boldsymbol{n},\boldsymbol{n}')\delta P(\boldsymbol{n}')$, i.e. $\delta P(\boldsymbol{n})$ is an eigenfunction of an $O(n,1)$-invariant integral operator $\hat{L}$ with kernel $L(\boldsymbol{n},\boldsymbol{n}')$. The space of such functions is spanned by the so-called spherical functions of group $O(n,1)$ (see, e.g. [12]). In our case of $O(n)$-symmetric $P(\theta)$ these are the zonal spherical functions

$$\psi_\nu(\theta) = (\sinh\theta)^{1-n/2} P^{-n/2+1}_{\nu+n/2-1}(\cosh\theta) \tag{21}$$

where $P^\mu_\nu(z)$ is the Legendre function. The function $\psi_\nu(\theta)$ is an eigenfunction of $\hat{L}$ with an eigenvalue $\Lambda(\nu)$ given by (20). Some properties of these functions are present in the appendix.

For the problem of stability of the asymptotic solution $P=1$, the relevant values of ν are real positive numbers. Indeed, the natural perturbation $\delta P(t)$ induced by the symmetry breaking term $D(t)\approx 1-\mathrm{e}^t$ in the region $t\ll 0$ is $\delta P(t) = \mathrm{e}^{\nu t}$ with $\nu = 1$. Analysis similar to that of [8] shows that if $m\Lambda_{\min} > 1$, where $\Lambda_{\min} = \min_\nu \Lambda(\nu)$, then the solution $P=1$ is unstable and collapses to the trival solution $P=0$ (ordered phase). This happens for any value of coupling constant J for $n\geqslant 1$ because in this case $K_{p+\nu}(J)\geqslant K_p(J)$ and $m\Lambda_{\min} = m\Lambda(0) = m > 1$. On the other hand, for $0\leqslant n<1$, $K_p(J) = K_{|p|}(J)$, and $p+\nu$ may be smaller than $|p|$. In this case there is a transition at the value of the coupling constant J_{c} determined by the equation

$$m\frac{K_0(J_{\mathrm{c}})}{K_{|p|}(J_{\mathrm{c}})} = 1. \tag{22}$$

Solution of this equation for $m=2$ is shown in figure 1.

6. Correlators in the disordered phase

From now on we restrict our attention to the case $0\leqslant n<1$. We will also assume that magnetic field is very small: $H\ll 1$. In this case we can approximate $P(0)$ by 1. Then for $J<J_{\mathrm{c}}$ the function

$$P(0)\approx\begin{cases}1 & \theta\ll\ln(2/H)\\ 0 & \theta\gg\ln(2/H)\end{cases}$$

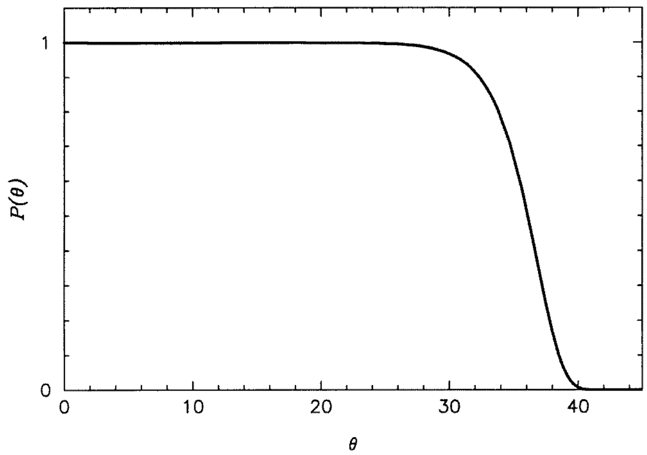

Figure 2. Function $P(\theta)$ for $n = 0$, $J = 10^{-6}$, $H = 10^{-10}$ obtained by numerical solution of equation (9).

with a kink connecting these two regions near $\theta = \ln(2/H)$. Typical solution of (9) of this type for $n = 0$, $J = 10^{-6}$, $H = 10^{-10}$ is shown in figure 2. Approximating this solution by a step function we estimate the partition function as

$$Z = aS_{n-1} \int_0^{\ln(2/H)} \mathrm{d}\theta \, \sinh^{n-1}\theta$$

which is not singular as $H \to 0$, so we will calculate it for $H = 0$

$$Z = aS_{n-1} \int_0^{\infty} \mathrm{d}\theta \, \sinh^{n-1}\theta = a\pi^p \Gamma(-p). \tag{23}$$

The finiteness of this quantity reflects the fact that the total volume of the hyperboloid $H^{n,1}$ is finite for $0 \leqslant n < 1$. We also obtain the 'order parameter'

$$\langle\sigma\rangle \approx \frac{aS_{n-1}}{Z} \int_0^{\ln(2/H)} \mathrm{d}\theta \, \sinh^{n-1}\theta \cosh\theta \approx \frac{aS_{n-1}}{nZ} H^{-n}. \tag{24}$$

So $\langle\sigma\rangle$ diverges when $H \to 0$ in the disordered phase, unless $n = 0$, in which case $\langle\sigma\rangle$ becomes non-critical (similar to the density of states in the Anderson transition).

To obtain the expression for correlators we again perform the change of variables $H\cosh\theta = \mathrm{e}^t$ in (6) and (7). To leading order in $1/H$, both G^{L} and G^{T}_{ij} become the same (up to a factor δ_{ij}/n)

$$G^{\mathrm{L}}(r) = \frac{aS_{n-1}N(r)}{Z} H^{-n-1} \int_{-\infty}^{\infty} \mathrm{d}t_0 \, \mathrm{e}^{nt_0} D(t_0) P^m(t_0) \left(\prod_{i-1}^{r} \int_{-\infty}^{\infty} \mathrm{d}t_i \, M_{\mathrm{L}}(t_{i-1}, t_i) \right) P(t_r)\, \mathrm{e}^{t_r}.$$

The range of integration for an operator with kernel $M_{\mathrm{L}}(t, t')$ is infinite, which means that the spectrum of eigenvalues is continuous. In this case (8) for the correlator $G(r)$ is

$$G^{\mathrm{L}}(r) = \frac{(m+1)aS_{n-1}}{mZ} H^{-n-1} \int_0^{\infty} \mathrm{d}\lambda \, W(\lambda) A^2(\lambda) (m\Lambda_\lambda)^r$$

where $A^2(\lambda) = \langle \mathrm{e}^{nt} | \phi^{\mathrm{r}}_\lambda(t)\rangle \langle \phi^{\mathrm{l}}_\lambda(t) | \, \mathrm{e}^t \rangle$ and $W(\lambda)$ comes from the normalization of eigenfunctions. Right and left eigenfunctions $\phi^{\mathrm{r,l}}_\lambda(t)$ in the limit $H \to 0$ should approach $\psi_{-p+\mathrm{i}\lambda}(\theta)$ and $\sinh^{n-1}\theta \, \psi_{-p+\mathrm{i}\lambda}(\theta)$, or in the region $t \ll 0$

$$\phi^{\mathrm{r,l}}_\lambda(t) \propto \frac{1}{\lambda} \exp(\mp pt) \sin(\lambda t + \delta(\lambda)) \tag{25}$$

with eigenvalue $\Lambda_\lambda = K_{i\lambda}(J)/K_p(J)$. As is suggested by (A.2) and (A.4) the normalization of $\psi^{r,l}_\lambda(t)$ should be

$$\int_{-\infty}^{\infty} dt\, \phi^l_\lambda(t)\phi^r_{\lambda'}(t) = \frac{\delta(\lambda - \lambda')}{W(\lambda)}$$

where $W(\lambda) \propto \lambda^2$ for small λ. The small λ behaviour of the 'phase shift' $\delta(\lambda)$ in (25) is found matching the asymptotic behaviour (25) to $\phi_\lambda(t) = 0$ for $t > 0$, which is again the effect of the term $D(t)$. Such matching gives that $\delta(\lambda)$ should be at least linear in λ for small λ. Using this we can show that $A(\lambda) \sim O(1)$ for small λ. Also expanding $m\Lambda_\lambda$ in $J_c - J$ and in λ we find $m\Lambda_\lambda \approx 1 - a_0(J_c - J) - a_2\lambda^2$. Combining all the above results we arrive at

$$G^L(r) \propto nH^{-n-1}\int_0^\infty d\lambda\, \lambda^2 \exp(-a_0(J_c - J)r - a_2\lambda^2 r) \propto nH^{-n-1}r^{-3/2}\exp(-r/\xi)$$
$$G^T_{ij} \propto \delta_{ij}H^{-n-1}r^{-3/2}\exp(-r/\xi)$$

where $\xi \sim (J_c - J)^{-1}$.

In the replica limit these equations reduce to $G^L(r) = 0$ and

$$\Gamma^T_{ij}(r) \equiv \lim_{H\to 0} HG^T_{ij}(r) \propto \delta_{ij}r^{-3/2}\exp(-r/\xi)$$

which has the same form as the density–density correlator in the localized regime in [7–9].

7. Correlators in the ordered phase close to the transition

In the case $J - J_c \ll J_c$ (just above the transition) we expect spontaneous symmetry breaking which, in terms of the function $P(\theta)$, takes place at some large scale A divergent at $J = J_c$ such that

$$P(\theta) \approx \begin{cases} 1 & \theta \ll \ln 2A \\ 0 & \theta \gg \ln 2A. \end{cases}$$

Similarly to the previous section we find that partition function is non-singular as $A \to \infty$, so we take it to be equal to its value at J_c (23). Again, as in (24), we find

$$\langle\sigma\rangle \approx \frac{aS_{n-1}}{nZ}A^n. \qquad (26)$$

The long-distance behaviour of the correlators in this phase is determined by the eigenvalues of the operator with kernel $M(\boldsymbol{n}, \boldsymbol{n}')$, see (2). Because of the form of the function $P(\theta)$ the integration range in θ for this operator is finite, and the spectrum of its eigenvalues is discrete. It is easy to see that for $H = 0$ the largest eigenvalue of this operator is $1/m$. Indeed, for $Q \in O(n, 1)$ using invariance of the kernel $L(\boldsymbol{n}, \boldsymbol{n}')$ and the measure $d\boldsymbol{n}$ we get $P(Q\boldsymbol{n}) = \int d\boldsymbol{n}'\, L(\boldsymbol{n}, \boldsymbol{n}')P^m(Q\boldsymbol{n}')$. Taking Q to be infinitesimally close to unity and expanding we obtain

$$\frac{dP(\boldsymbol{n})}{d\sigma}\boldsymbol{\pi} = m\int d\boldsymbol{n}'\, L(\boldsymbol{n}, \boldsymbol{n}')P^{m-1}(\boldsymbol{n}')\frac{dP(\boldsymbol{n}')}{d\sigma'}\boldsymbol{\pi}'. \qquad (27)$$

The functions $|f_i\rangle = \pi_i dP(\boldsymbol{n})/d\sigma$ are the Goldstone modes associated with symmetry breaking. Integrating out angular variables in (27) we have

$$\frac{dP(\theta)}{d\theta} = m\int_0^\infty d\theta'\, L_T(\theta, \theta')P^{m-1}(\theta')\frac{dP(\theta')}{d\theta'}. \qquad (28)$$

In the asymptotic region $1 \ll \theta,\ \theta' \ll \ln 2A$ the kernel $L_{\rm T}(\theta,\theta') \to L(\theta-\theta')$ of (18), equation (28) becomes the same as (19) which means that $P'(\theta)$ has the asymptotic behaviour

$$\frac{\mathrm{d}P(\theta)}{\mathrm{d}\theta} \approx \frac{C}{\lambda}\ \exp(-p\theta)\sin\lambda\theta \tag{29}$$

where λ satisfies $K_{\mathrm{i}\lambda}(J)/K_p(J) = 1/m$. Expanding this in small λ and small $J - J_{\rm c}$ we find

$$\lambda \sim (J - J_{\rm c})^{1/2}. \tag{30}$$

The fast decrease of $P^{m-1}(\theta)$ near $\theta = \ln 2A$ chooses the value of λ such that function (29) has the first node at this point: $\lambda = \pi/\ln 2A$. Combining this with (30) we find

$$A \sim \exp(\text{constant} \times (J - J_{\rm c})^{-1/2}). \tag{31}$$

The value of the constant C in (29) may be found as follows. $\mathrm{d}P/\mathrm{d}\theta$ is related to the function $\delta P(\theta)$ of (19) in an obvious manner: $\mathrm{d}P/\mathrm{d}\theta = -\mathrm{d}\delta P/\mathrm{d}\theta$. Using this and writing $\delta P(\theta) \approx C_1\lambda^{-1}\exp(-p\theta)\sin(\lambda\theta + \delta(\lambda))$ we get $\delta(\lambda) = \tan^{-1}(\lambda/p) \approx \lambda/p$ and $C = -C_1(\lambda^2 + p^2)^{1/2} \approx -C_1|p|$. The function δP should approach the value unity near $\theta = \ln 2A$. This gives $C_1 \approx |p|(2A)^p$ and

$$C \approx -p^2(2A)^p. \tag{32}$$

Now we evaluate the largest eigenvalue $\Lambda_{\max}$ of the operator with kernel $M(\boldsymbol{n},\boldsymbol{n}')$ for small H using first-order perturbation theory: $\Lambda_{\max} = \langle f_i|\hat{M}|f_i\rangle/\langle f_i|f_i\rangle$ (no summation!). Since kernel $M(\boldsymbol{n},\boldsymbol{n}')$ is non-symmetric, $\langle f_i|$ differs from $|f_i\rangle$ by the factor $P^{m-1}(\boldsymbol{n})$. Then we have

$$\begin{aligned}\langle f_1|f_1\rangle &= \int \mathrm{d}\boldsymbol{n}\, P^{m-1}(\boldsymbol{n})\left(\frac{\mathrm{d}P}{\mathrm{d}\sigma}\pi_1\right)^2 \\ &= \frac{aS_{n-1}}{n}\int_0^\infty \mathrm{d}\theta\ \sinh^{n-1}\theta\, P^{m-1}(\theta)\left(\frac{\mathrm{d}P(\theta)}{\mathrm{d}\theta}\right)^2.\end{aligned}$$

Using (29) and (32) we estimate $\langle f_1|f_1\rangle$ to be

$$\langle f_1|f_1\rangle \propto \frac{aS_{n-1}}{n}\frac{C^2}{\lambda^2}\int_0^{\ln 2A}\mathrm{d}\theta\ \sin^2\lambda\theta \propto \frac{aS_{n-1}}{n}A^{n-1}\ln^3 2A.$$

In the presence of H, (27) is replaced by

$$\frac{\mathrm{d}P(\boldsymbol{n})}{\mathrm{d}\sigma}\boldsymbol{\pi} = \int \mathrm{d}\boldsymbol{n}'\, L(\boldsymbol{n},\boldsymbol{n}')\frac{\mathrm{d}}{\mathrm{d}\sigma'}(P^m(\boldsymbol{n}')\,\mathrm{e}^{-H\sigma'})\boldsymbol{\pi}'.$$

Using this equation we perform integration by parts and keep only linear terms in H to find

$$\langle f_1|\hat{M}|f_1\rangle = \frac{\langle f_1|f_1\rangle}{m} - H\frac{Z\langle\sigma\rangle}{m^2(m+1)}.$$

Then for the maximal eigenvalue we have $m\Lambda_{\max} = 1 - H/((m-1)\rho_{\rm s})$, where the 'spin stiffness'

$$\rho_{\rm s} = \frac{m(m+1)}{(m-1)}\frac{\langle f_1|f_1\rangle}{Z\langle\sigma\rangle} \propto A^{-1}\ln^3 2A \sim (J-J_{\rm c})^{-3/2}\exp(-\text{constant}\times(J-J_{\rm c})^{-1/2}). \tag{33}$$

For the final evaluation of the correlator $\langle A(\boldsymbol{n}_0)B(\boldsymbol{n}_r)\rangle_\omega$ we have to calculate $A^2(\Lambda) = \sum_{i=1}^n\langle A|f_i\rangle\langle f_i|B\rangle/\langle f_1|f_1\rangle$. For the longitudinal correlator $A(\boldsymbol{n}) = B(\boldsymbol{n}) = \cosh\theta$, in which case $\langle f_i|\cosh\theta\rangle = \int \mathrm{d}\boldsymbol{n}\, P^m(\boldsymbol{n})\sigma\pi_i(\mathrm{d}P(\boldsymbol{n})/\mathrm{d}\sigma) = 0$ and $A^2(\Lambda) = 0$. This means that the decay of $G^{\rm L}(r)$ is governed by smaller eigenvalues and, therefore, $G^{\rm L}(r)$ is massive even for $H = 0$, which is to be compared with (17). On the other hand, for Goldstone modes we have $\langle f_i | \pi_j \rangle = \int \mathrm{d}\boldsymbol{n}\, P^m(\boldsymbol{n}) \pi_i \pi_j (\mathrm{d}P(\boldsymbol{n})/\mathrm{d}\sigma) = -\delta_{ij} Z \langle \sigma \rangle / (m+1)$ and for the transverse correlator we finally have

$$
\begin{aligned}
G^{\mathrm{T}}_{ij}(r) &= \delta_{ij} \frac{m+1}{mZ} A^2(\Lambda)(m\Lambda_{\max})^r \\
&= \frac{\langle \sigma \rangle}{(m-1)\rho_{\mathrm{s}}} \exp\left(-\frac{H}{(m-1)\rho_{\mathrm{s}}} r \right). \qquad (34)
\end{aligned}
$$

Note that again, as before, in the replica limit our equations (31), (33) and (34) exactly correspond to the results of [7–9].

8. Discussion

In this paper, we have solved the non-compact $O(n, 1)$ model on the Bethe lattice for arbitrary n. The analytical continuation procedure allows us to consider n as being an arbitrary positive number. We find that for $n > 1$ the symmetry of the model is always broken, so that the system is ordered and the order parameter has a finite value. This is an agreement with [13], where the system was shown to be ordered for $n = 1$ and $n \to \infty$ above two dimensions. (Note that the Bethe lattice is effectively infinite dimensional.) However, for $0 < n < 1$ we find a transition from the ordered to disordered phase, when the coupling strength J decreases. In the replica limit $n \to 0$, our solution reproduces exactly that for the supersymmetric version of the model. The latter is nothing but the hyperbolic superplane introduced by Zirnbauer [5, 6] as a toy model for the Anderson localization transition. In the whole region $0 \leqslant n < 1$ the qualitative picture of the transition and the critical behaviour are analogous to those of the supersymmetric model of Zirnbauer, which shows in turn all the essential features of the Anderson transition on the Bethe lattice studied in [7–9]. The success of the replica trick may seem surprising, since we know [4] that it gives wrong results in the case of the level correlation problem. The crucial difference is that near the Anderson transition only the non-compact sector of the supersymmetric σ-models is essential, whereas for the level correlation problem both compact and non-compact sectors are equally important. The similar reason explains the agreement between the replica trick renormalization group calculation of asymptotics of various distributions [14] and the recent study of these asymptotics via the supersymmetry method [15].

When the manuscript was in preparation, we learnt about the work by Dupré [16] who studied the supersymmetric model of Zirnbauer [5, 6] in three dimensions and found the critical behaviour analogous to that expected for the Anderson localization transition.

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Appendix

The functions (21) have the following asymptotic behaviour

$$\psi_\nu(\theta \gg 1) \approx \frac{2^{n/2-1}}{\sqrt{\pi}} \left(\frac{\Gamma(\nu+p)}{\Gamma(\nu+n-1)} \, \mathrm{e}^{\nu\theta} + \frac{\Gamma(-\nu-p)}{\Gamma(-\nu)} \, \mathrm{e}^{-(\nu+n-1)\theta} \right). \quad \text{(A.1)}$$

Among the functions (21) there is a special subset with $\nu = -p + \mathrm{i}\lambda$, $\lambda \geqslant 0$. These functions are real and form a continuous basis in the space $L^2([1, \infty), \mathrm{d}\theta \ \sinh^{n-1}\theta)$:

$$\int_0^\infty \mathrm{d}\theta \ \sinh^{n-1}\theta \, \psi_{-p+\mathrm{i}\lambda}(\theta)\psi_{-p+\mathrm{i}\lambda'}(\theta) = \frac{1}{\mu(\lambda)} \delta(\lambda - \lambda') \quad \text{(A.2)}$$

$$\int_0^\infty \mathrm{d}\lambda \, \mu(\lambda)\psi_{-p+\mathrm{i}\lambda}(\theta)\psi_{-p+\mathrm{i}\lambda}(\theta') = \sinh^{1-n}\theta \, \delta(\theta - \theta')$$

$$\mu(\lambda) = \left| \frac{\Gamma(p+\mathrm{i}\lambda)}{\Gamma(\mathrm{i}\lambda)} \right|^2 . \quad \text{(A.3)}$$

For $n \neq 1$ and for small values of λ, the asymptotics (A.1) and the spectral measure (A.3) take the form

$$\psi_{-p+\mathrm{i}\lambda}(\theta) \approx \frac{2^{n/2}}{\sqrt{\pi}\,\Gamma(p)\lambda} \exp(-p\theta) \sin \lambda\theta \qquad \mu(\lambda) \approx \Gamma^2(p)\lambda^2. \quad \text{(A.4)}$$

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